Simple Harmonic Motion : SHM

10 TH AUGUST 2020

A simple harmonic oscillator : simple pendulum

(a)

$$m\ddot{x} + mg\,\frac{x}{l} = 0$$

$$ml\ddot{\theta} + mg\,\theta = 0$$

$$\omega^2 = g/l$$

θ

l

$\leftarrow x \rightarrow$

m

mg

$mg\sin\theta \approx mg\,\theta$

$$\approx mg\,\frac{x}{l}$$

Small angle approximation is applied.
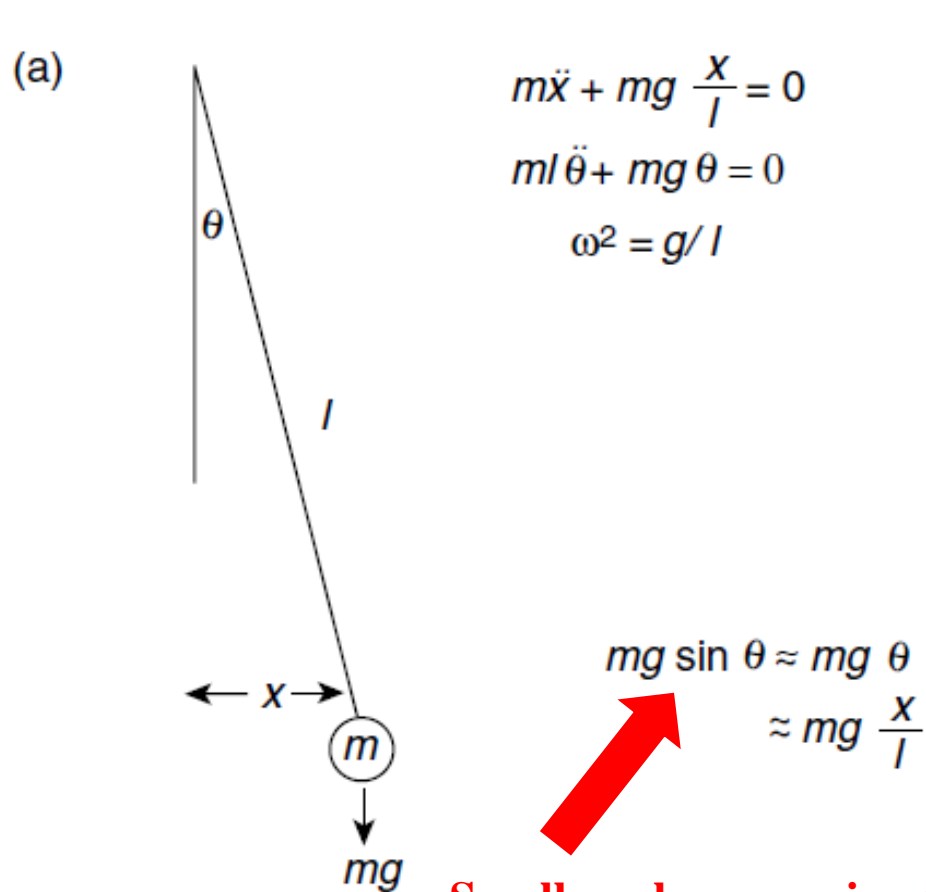
- (Guessed) Solutions of the equation of motion

$$\theta_1 = A\cos\omega t; \theta_2 = B\sin\omega t$$

where A, B = arbitrary constants and $\omega = (g/l)^{1/2}$

- The combination; $\theta_1 + \theta_2$, becomes the general solution and can be written as

$$\theta = \theta_1 + \theta_2 = a\sin(\omega t + \phi)$$

where $A = a\sin\phi; B = a\cos\phi; \phi = $ initial phase

A pendulum with a large oscillating angle

- Starting with the original equation of motion of a pendulum $\quad \ddot{\theta} + \dfrac{g}{l}\sin\theta = 0$

- Multiply both sides of the equation by $\dot{\theta}$ and follow by integrating the modified equation,

$$\dot{\theta}^2 = \frac{2g}{l}\cos\theta + const.$$

- Assume the initial conditions as follows: $\theta(t=0) = \alpha$ and $\dot{\theta}(t=0) = 0$ where α is an initial angular position of the pendulum.

- The equation becomes $\quad\quad \dot{\theta}^2 = \dfrac{2g}{l}(\cos\theta - \cos\alpha)$

- We then integrate this equation again to determine a period T_α (time taken for the pendulum to swing from $+\alpha$ to $-\alpha$ and back to $+\alpha$)

$$\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{2g}{l}}\,\frac{T_\alpha}{4}$$

Complete elliptic integral of the 1ˢᵗ kind

•We can transform $\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{2g}{l}}\frac{T_\alpha}{4}$ into $T_\alpha = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2 sin^2\varphi}}$ where $k = sin\frac{\alpha}{2}$

which clearly show that the oscillating period depends on the angular position α and the length of the string l.

•The derived expression corresponds to the complete elliptic integral of the 1ˢᵗ kind: $K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}}$, therefore period T_α can be written as

$$T_\alpha = 4\sqrt{\frac{l}{g}}K\left(sin\frac{\alpha}{2}\right)$$

- Consider $\alpha < 90^0, \frac{1}{2}\alpha < 45^0, \therefore \left(sin\frac{\alpha}{2}\right)^2 < \frac{1}{2}.$ This gives $k\sin\varphi < 1$.

What if the small angle approximation is applied to this expression!
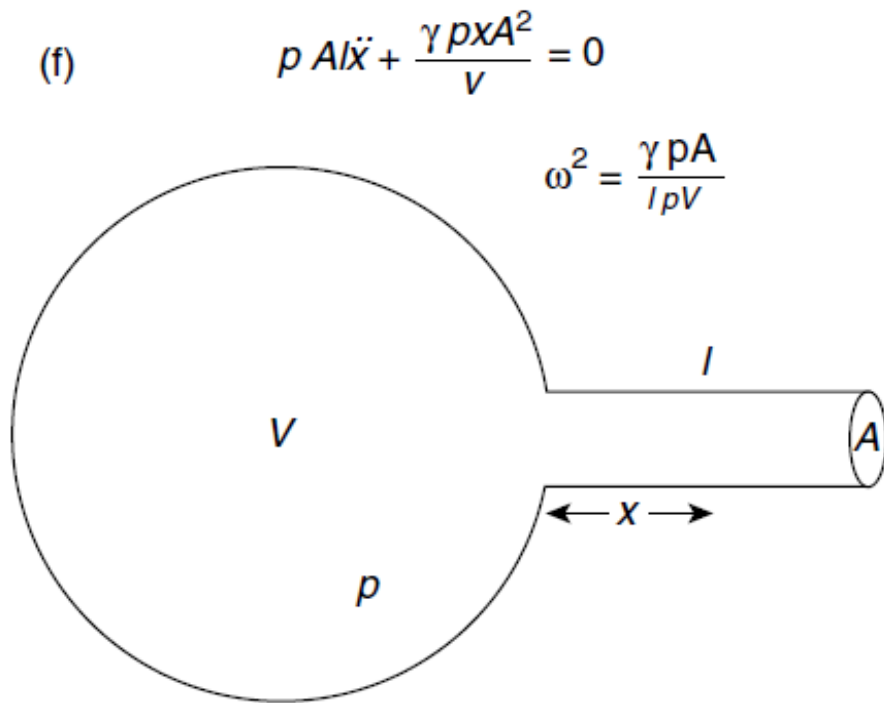
- $\frac{1}{\sqrt{1-k^2 sin^2\varphi}}$ can be binomial expanded and we end up with

$$T_\alpha = 4\sqrt{\frac{l}{g}}\frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2\left(sin\frac{\alpha}{2}\right)^2 + \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\left(sin\frac{\alpha}{2}\right)^4 + \cdots\right)$$
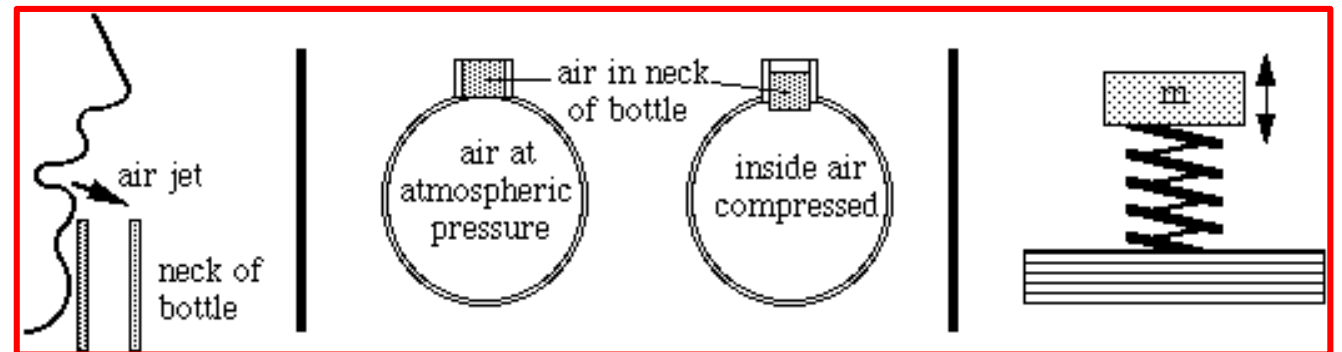
Example : Helmholtz Oscillator

(f)
$$p\,Al\ddot{x} + \frac{\gamma\,pxA^2}{V} = 0$$

$$\omega^2 = \frac{\gamma\,pA}{l\,pV}$$

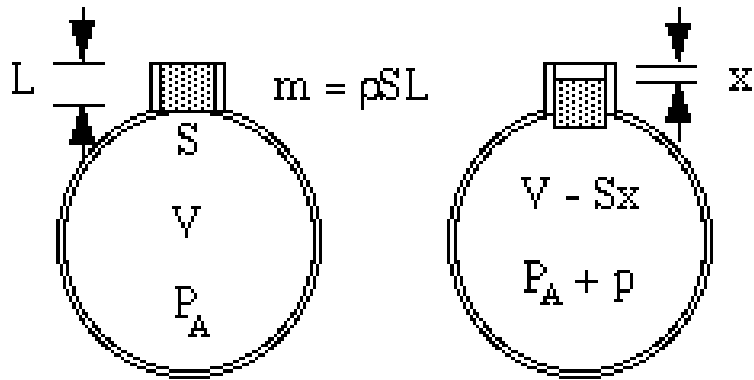https://newt.phys.unsw.edu.au/jw/Helmholtz.html

- A Helmholtz resonator or Helmholtz oscillator is a container of gas (usually air) with an open hole (or neck or port).

- The vibrating part is a volume of air in and near the open hole because of the 'springiness' of the air inside.

Analysis the Helmholtz oscillator (1)

L = effective length of air in the neck
S = cross sectional area
V = initial air volume
P_A = atmospheric pressure
ρ = air density
p = increasing pressure

- The displacement of air column is given as x.

- The change of air volume is Sx.

- The change of pressure is p.

- This creates an unbalanced force, pS, between the top and bottom of the neck.

- Under the adiabatic process : PV^γ = constant.

$$\therefore dP/P = -\gamma dV/V$$

- Therefore, $p/P_A = -\gamma Sx/V$

Analysis the Helmholtz oscillator (2)

Due to the pressure difference between the top and bottom of the neck, the motion of the air mass inside the neck follows the Newton's laws of motion.
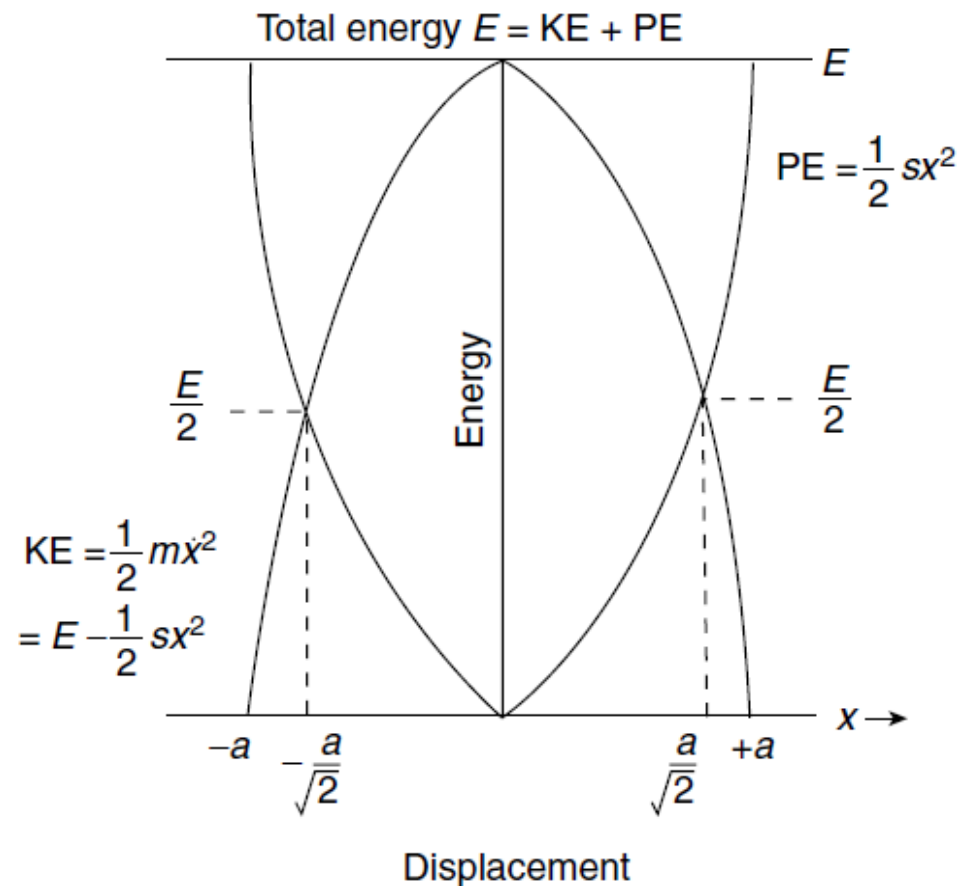
Initial phase

- **Initial phase** or **phase constant** is the phase at t = 0.

- The knowledge of the phase constant enables us to find out how far from the mean position the system was at time t = 0.

- The phase constant ϕ are determined from the initial conditions.

Velocity and Acceleration in SHM
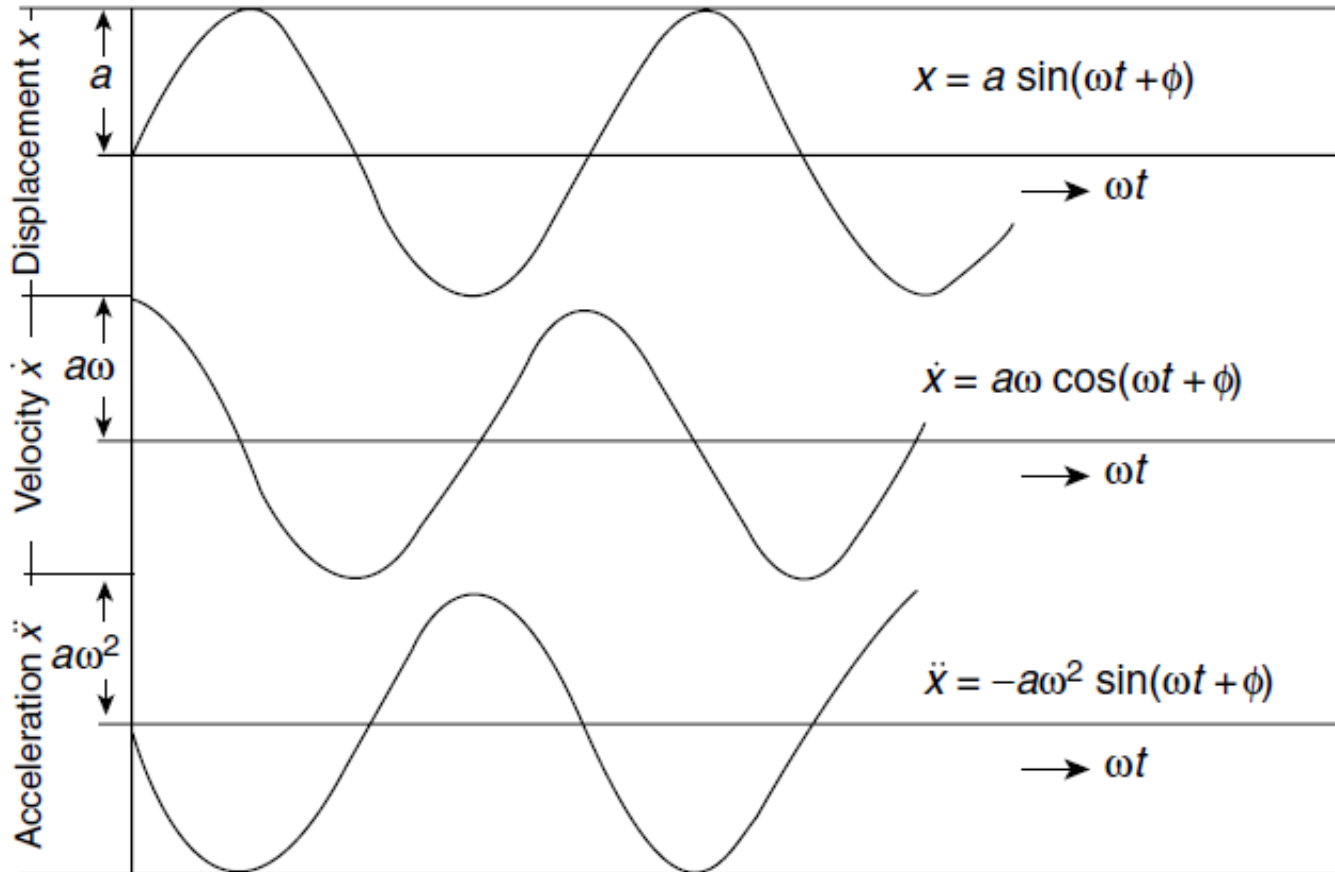
- Displacement lags velocity by $\pi/2$ rad and is π rad out of phase with the acceleration.

- The initial phase constant ϕ taken as zero.

- a = amplitude

  $a\omega$ = velocity amplitude

  $a\omega^2$ = acceleration amplitude

Energy of a Simple Harmonic Oscillator

Total energy E = KE + PE

$PE = \frac{1}{2} sx^2$

$\frac{E}{2}$

Energy

$\frac{E}{2}$

$KE = \frac{1}{2} m\dot{x}^2$

$= E - \frac{1}{2} sx^2$

$-a$ $-\frac{a}{\sqrt{2}}$ $\frac{a}{\sqrt{2}}$ $+a$ $x \rightarrow$
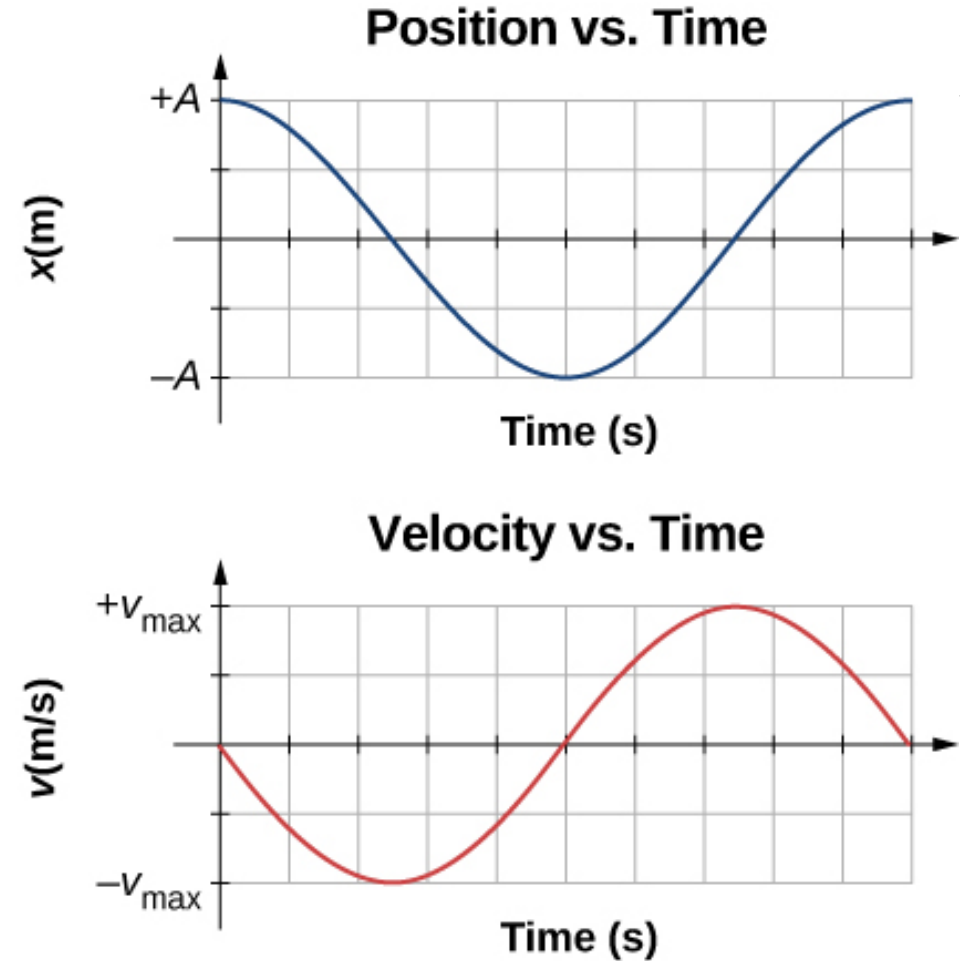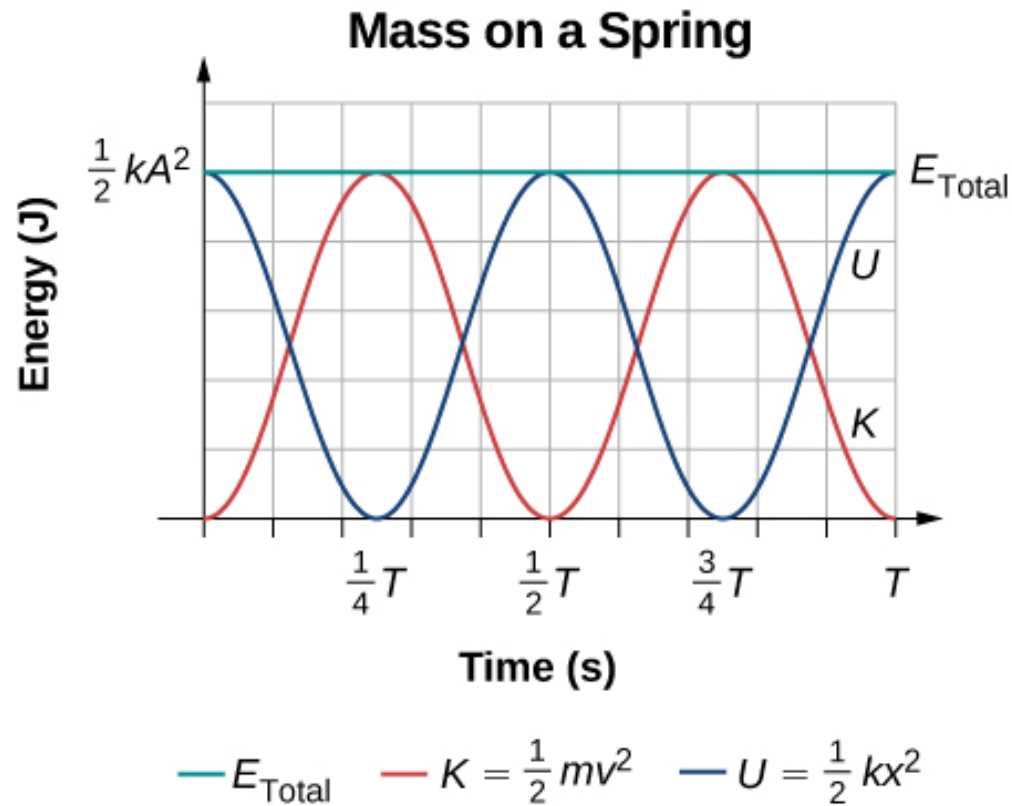
Displacement

- The solution $x = a \sin(\omega t + \phi)$ reveals that the total energy of the oscillator must remain constant because the maximum displacement is regained after every half cycle.

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2\right) = 0$$

$$\therefore m\ddot{x} + sx = 0; s = \text{stiffness}$$

Graph of the kinetic energy, potential energy, and total energy of a block oscillating on a spring in SHM vs time

Mass on a Spring

Energy (J)

$\frac{1}{2}kA^2$

E_{Total}

U

K

$\frac{1}{4}T$ $\quad$ $\frac{1}{2}T$ $\quad$ $\frac{3}{4}T$ $\quad$ T

Time (s)

— E_{Total} $\quad$ — $K = \frac{1}{2}mv^2$ $\quad$ — $U = \frac{1}{2}kx^2$

Position vs. Time

$+A$

$x(m)$

$-A$

Time (s)

Velocity vs. Time

$+v_{max}$

$v(m/s)$

$-v_{max}$

Time (s)

Example : Equation of motion by energy equation

- The general form for the total energy of SHM is

    $E = (1/2)$ mass (velocity)^2 $+$ $(1/2)$ stiffness (displacement)^2
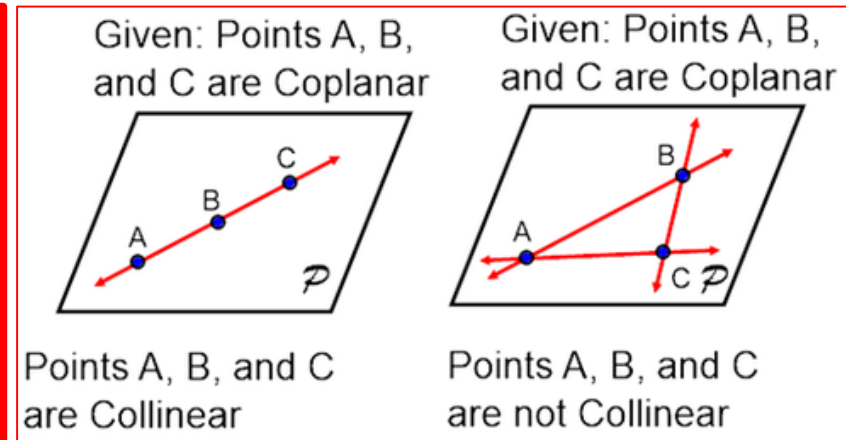
- **For the Helmholtz oscillator :**

- Mass of the air column = ……………… is displaced from equilibrium position by a distance of x.

- The pressure change p =…………

- The stiffness of the system is given by ………………………

- This equation of motion is found to be ………………………

Superposition of Harmonic Oscillations

- The **superposition principle** states that " The resultant of two or more harmonic displacements is simply the algebraic sum of the individual displacement".

- The superposition principle holds only for linear differential equations, i.e.

   a small oscillation with the equation of motion : $\ddot{x} = -\omega^2 x$

- Consider superposition of

    -Two simple harmonic vibrations in one dimension,

     (Two **collinear** harmonic oscillations)

    - Two perpendicular simple harmonic vibrations,

Given: Points A, B, and C are Coplanar

Points A, B, and C are Collinear

Given: Points A, B, and C are Coplanar

Points A, B, and C are not Collinear

https://www.quora.com/Is-it-true-that-if-three-points-are-coplanar-they-are-collinear

Two Simple Harmonic Vibrations in One Dimension (1)

- Two simple harmonic oscillations of **equal angular frequencies** but of different amplitudes and phases are given by

$$x_1 = a_1 \cos\left(\omega t + \phi_1\right), x_2 = a_2 \cos\left(\omega t + \phi_2\right)$$

- The algebraic sum of the two displacement is found to be

$$x = x_1 + x_2 = R\cos\left(\omega t + \theta\right)$$

- Note : the resulting motion is simple harmonic with an angular frequency ω, the same as that of the individual SHMs. The resulting motion has amplitude R and a phase constant θ.

Two Simple Harmonic Vibrations in One Dimension (2)

- The resulting displacement can also be found from the addition of vectors.

- By using the cosine law, the resulting amplitude R is given as

$$R^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos \delta$$

Consider at t = 0
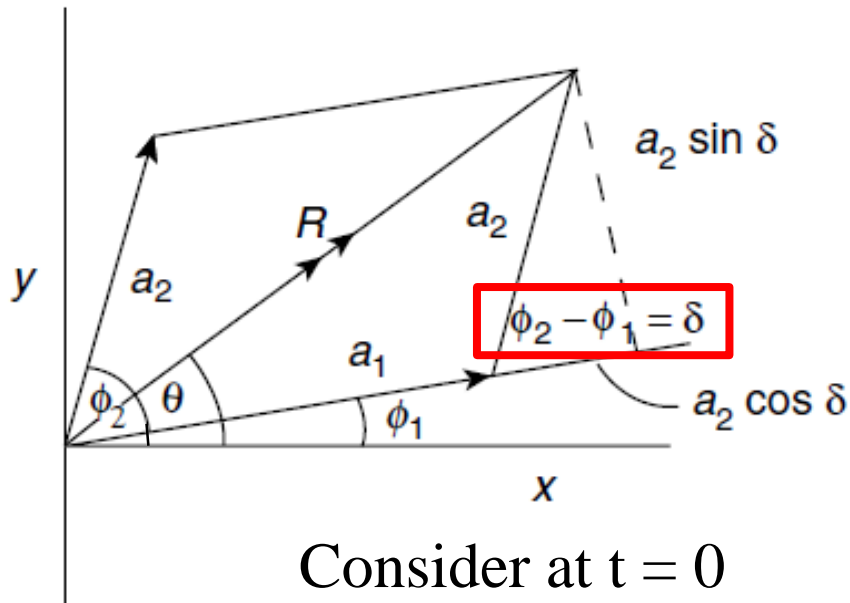
The phase constant of R is found to be

$$\tan \theta = \frac{a_1 \sin \phi_1 + a_2 \sin \phi_2}{a_1 \cos \phi_1 + a_2 \cos \phi_2}$$

Two Simple Harmonic Vibrations in One Dimension (3)

• Alternatively, the algebraic sum of the two harmonic functions can be written as

$$x = x_1 + x_2 = a_1 \cos(\omega t + \phi_1) + a_2 \cos(\omega t + \phi_2) = R \cos(\omega t + \theta)$$

$$R \cos(\omega t + \theta) = (a_1 \cos\phi_1 + a_2 \cos\phi_2)\cos\omega t - (a_1 \sin\phi_1 + a_2 \sin\phi_2)\sin\omega t$$

define $\quad R \cos\theta = a_1 \cos\phi_1 + a_2 \cos\phi_2$

$$R \sin\theta = a_1 \sin\phi_1 + a_2 \sin\phi_2$$

This also gives $\quad R^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_2 - \phi_1)$

$$= a_1^2 + a_2^2 + 2a_1 a_2 \cos\delta$$

and $\qquad\qquad \tan\theta = \dfrac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2}$
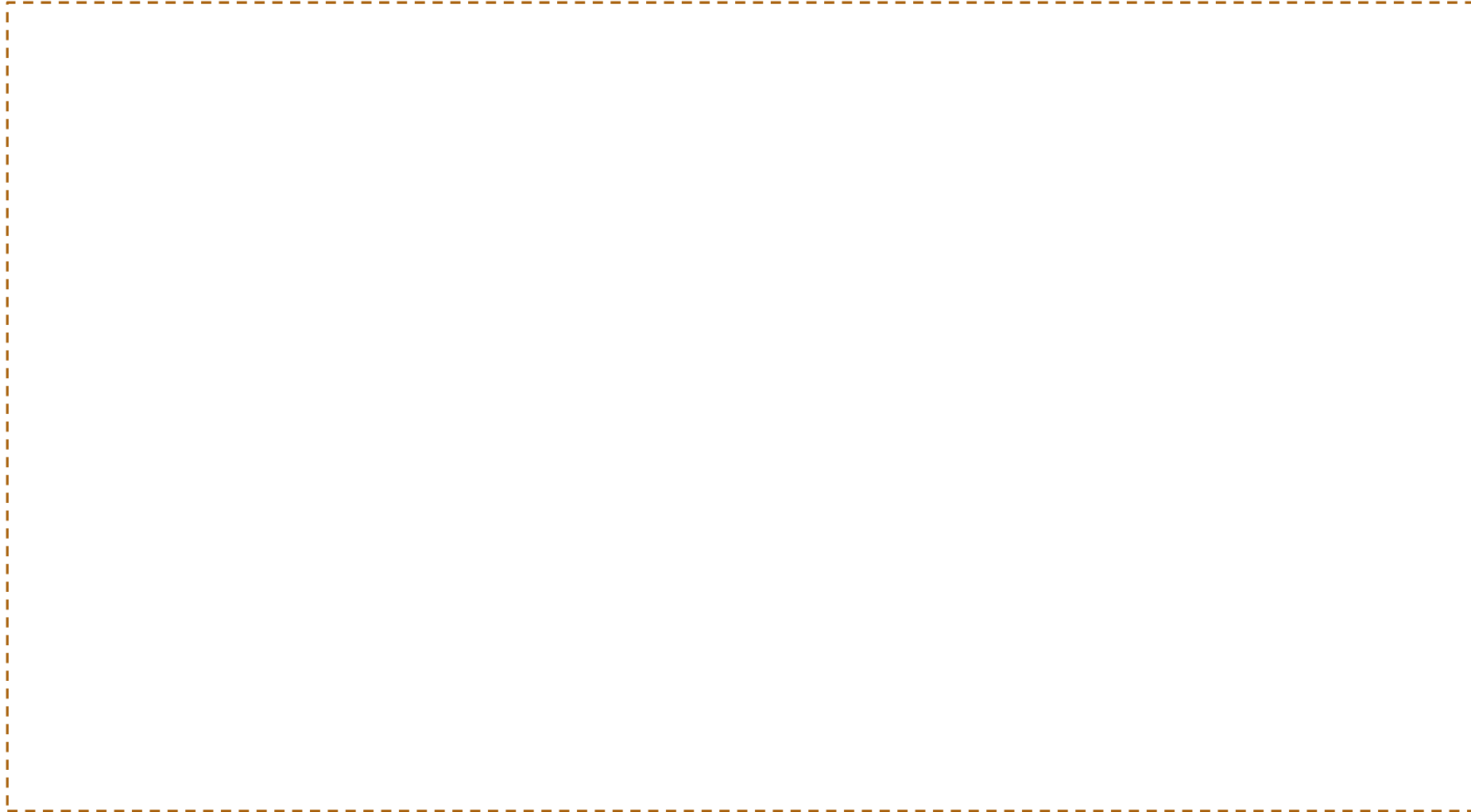
Example

The intensity of the pattern when light from two slits interferes will be seen to depend on the superposition of two simple harmonic oscillations of equal amplitude and phase difference δ. If intensity I = total amplitude squared.

Show that the **maximum** intensity is given by

$$I_{max} \propto \cos^2 \frac{\delta}{2}$$

Suppose the electric field from each slit is given as $E_1 = a\cos(\omega t)$ and $E_2 = a\cos(\omega t + \delta)$

Two Simple Harmonic Vibrations in One Dimension (4)

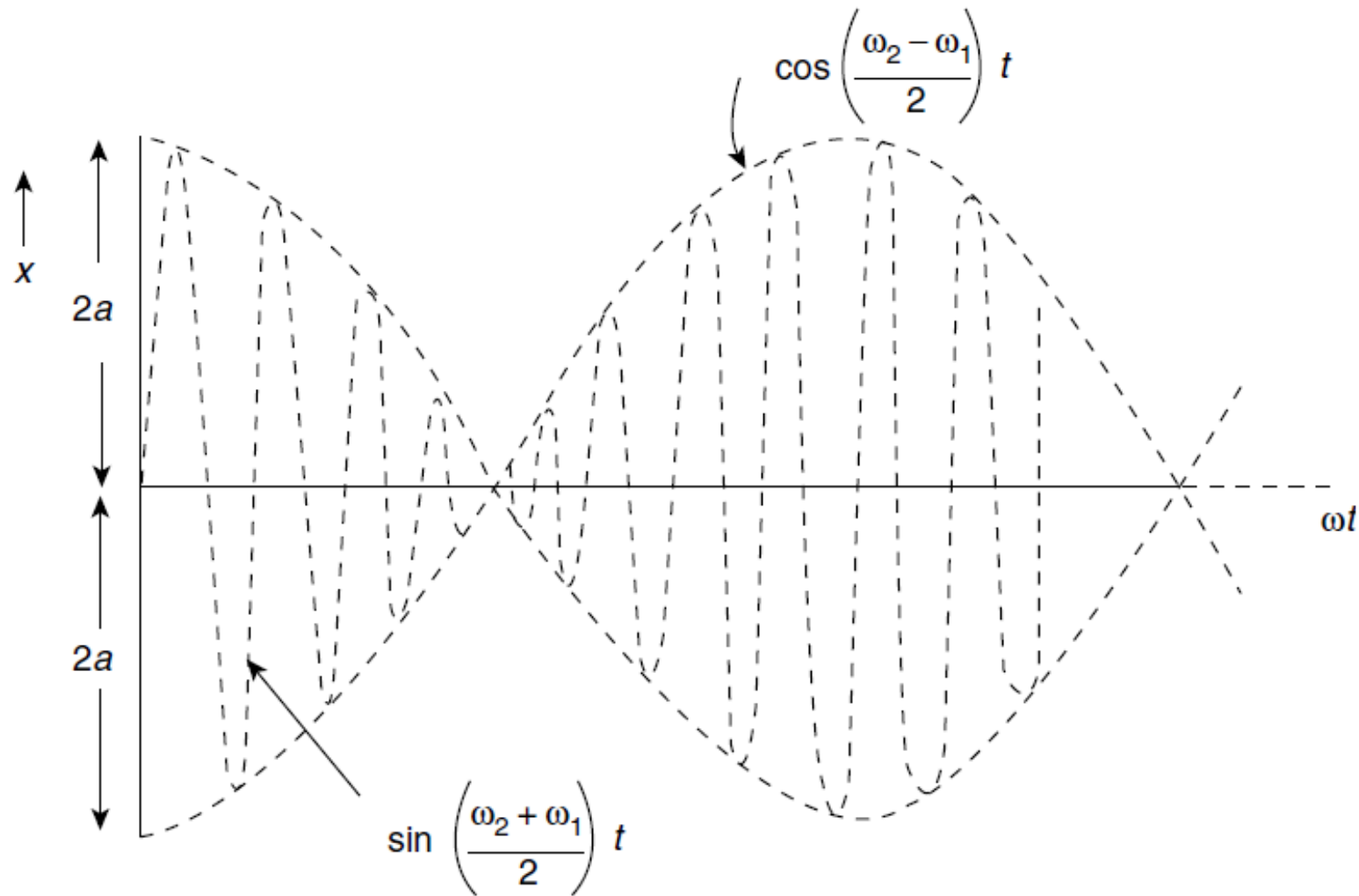- Consider two vibrations of equal amplitudes but different frequencies,

$$x_1 = a \sin \omega_1 t, \; x_2 = a \sin \omega_2 t; \; \omega_2 > \omega_1$$

- The resulting displacement is given by

$$x = x_1 + x_2 = 2a \sin \frac{(\omega_1 + \omega_2)t}{2} \cos \frac{(\omega_2 - \omega_1)t}{2}$$

- $\omega_2 - \omega_1$ is the beat frequency.

Superposition of two simple harmonic displacements with slightly different frequencies

$$\cos\left(\frac{\omega_2-\omega_1}{2}\right)t$$

$$\sin\left(\frac{\omega_2+\omega_1}{2}\right)t$$

The displacement graph shows the amplitude modulation of the high frequency components

Superposition of two perpendicular simple harmonic vibrations (1)

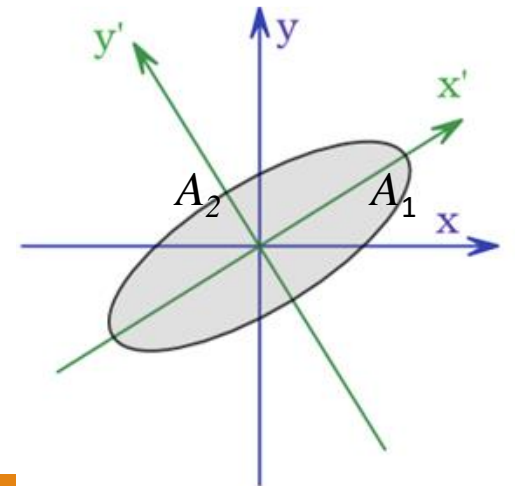**(1) Vibrations Having Equal Frequencies**

   Two displacements in x and y axis

$$x = A_1 \cos(\omega t + \alpha_1); \; y = A_2 \cos(\omega t + \alpha_2)$$

An expression, **which is the general equation for an ellipse**, involving only x and y and the constants ϕ_1 and ϕ_2

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} \cos\delta = \sin^2\delta; \; \delta = \alpha_2 - \alpha_1$$

(derive this equation!) A$_1$ and A$_2$ = semi major and semi minor axes
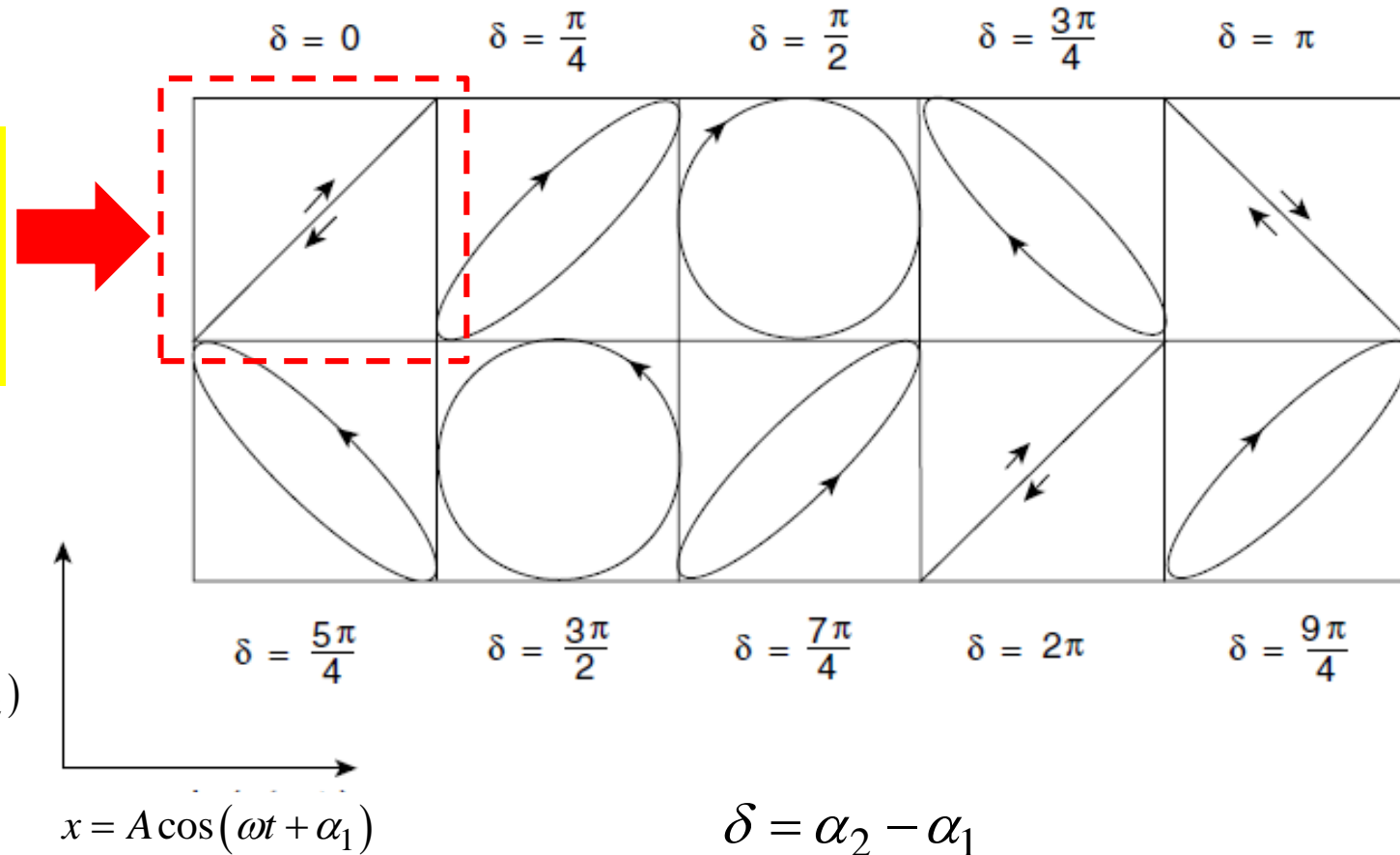
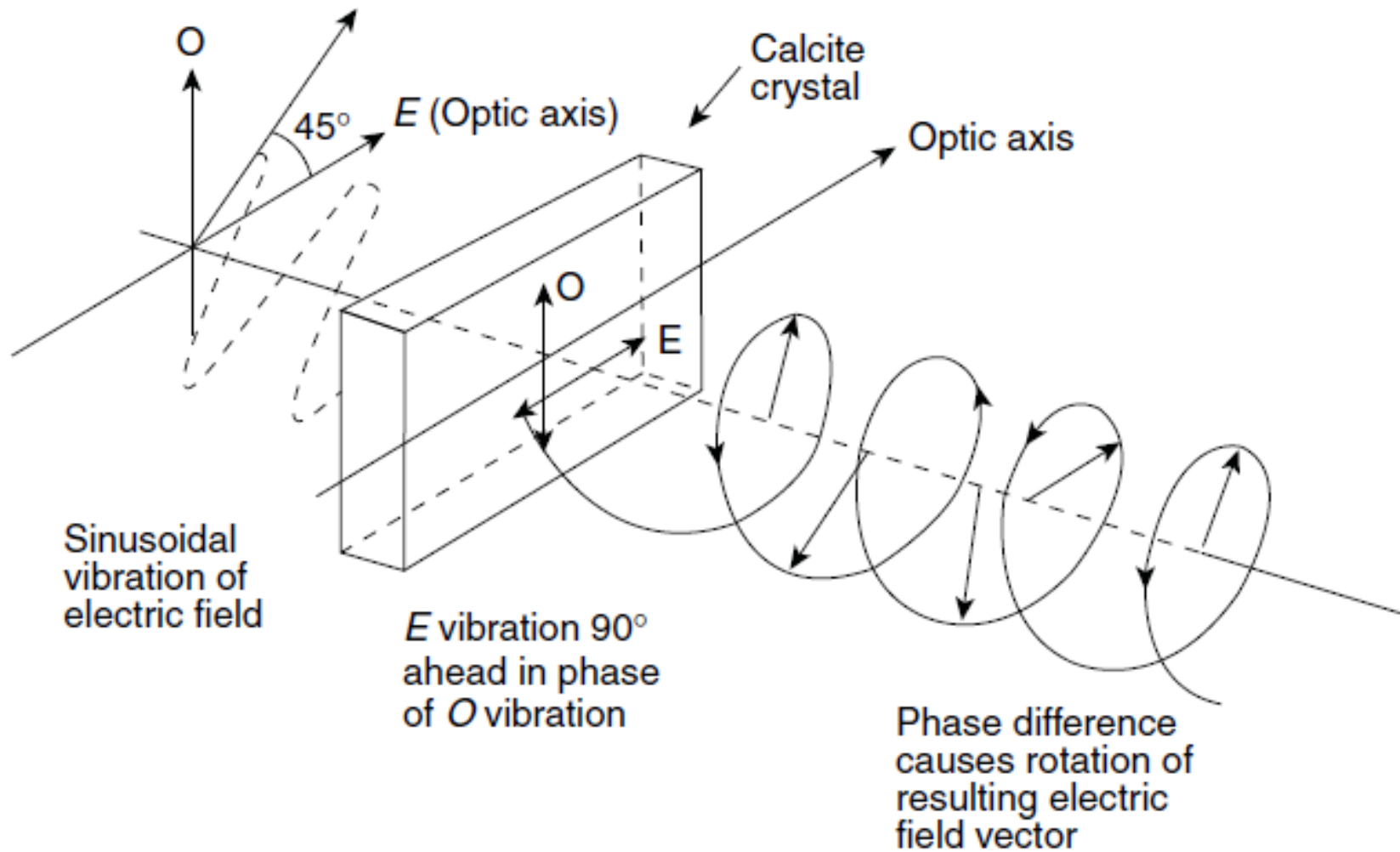# Superposition of two perpendicular simple harmonic vibrations (2)

Paths traced by a system vibrating simultaneously in two perpendicular directions with simple harmonic motions of equal frequency.

$$\delta = 0; \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} = 0$$
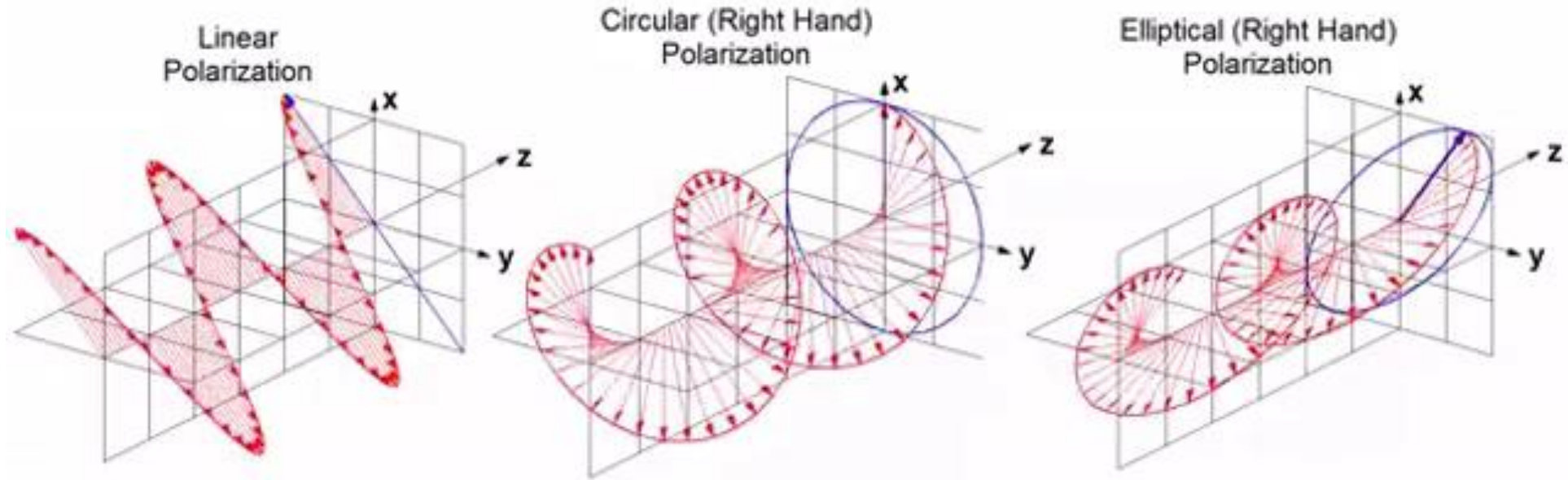
$$\therefore y = \frac{A_2}{A_1} x$$

How to check the handedness?

$\delta = 0$ $\delta = \frac{\pi}{4}$ $\delta = \frac{\pi}{2}$ $\delta = \frac{3\pi}{4}$ $\delta = \pi$

$\delta = \frac{5\pi}{4}$ $\delta = \frac{3\pi}{2}$ $\delta = \frac{7\pi}{4}$ $\delta = 2\pi$ $\delta = \frac{9\pi}{4}$

$$y = A\cos(\omega t + \alpha_2)$$

$$x = A\cos(\omega t + \alpha_1)$$

$$\delta = \alpha_2 - \alpha_1$$

The polarization transformation due to a linear light passing through a birefringence crystal can be determined from the superposition of two perpendicular harmonic vibrations. The final output polarization is a circularly polarized.

Polarization states

Superposition of two perpendicular simple harmonic vibrations (3)

(2) Vibrations Having Different Frequencies (Lissajous Figures)
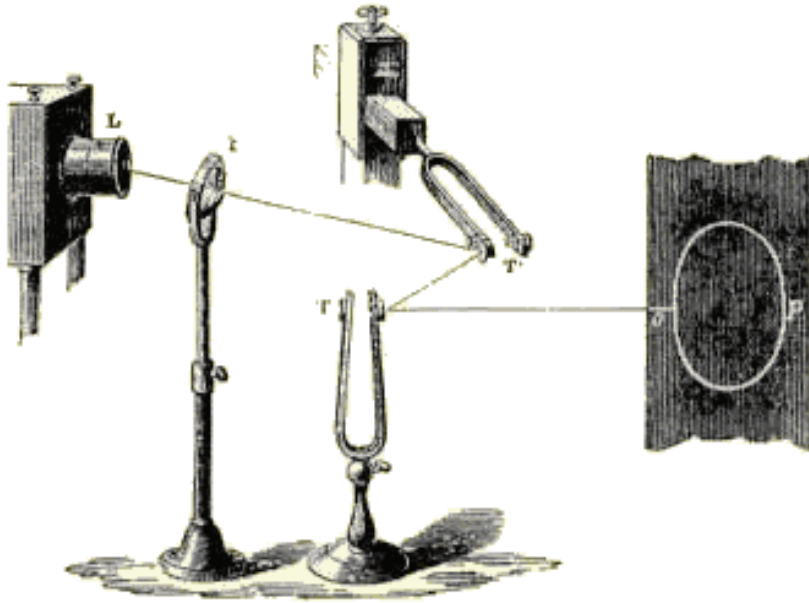
Lissajous' method of calibrating one tuning fork against another. When the two forks gave the same frequency, a beam reflected off both forks and projected on a screen, would form a steady ellipse.

An experimental setup made by Jules Antoine Lissajous to study the superposition of two perpendicular vibrations with **different angular frequencies**.
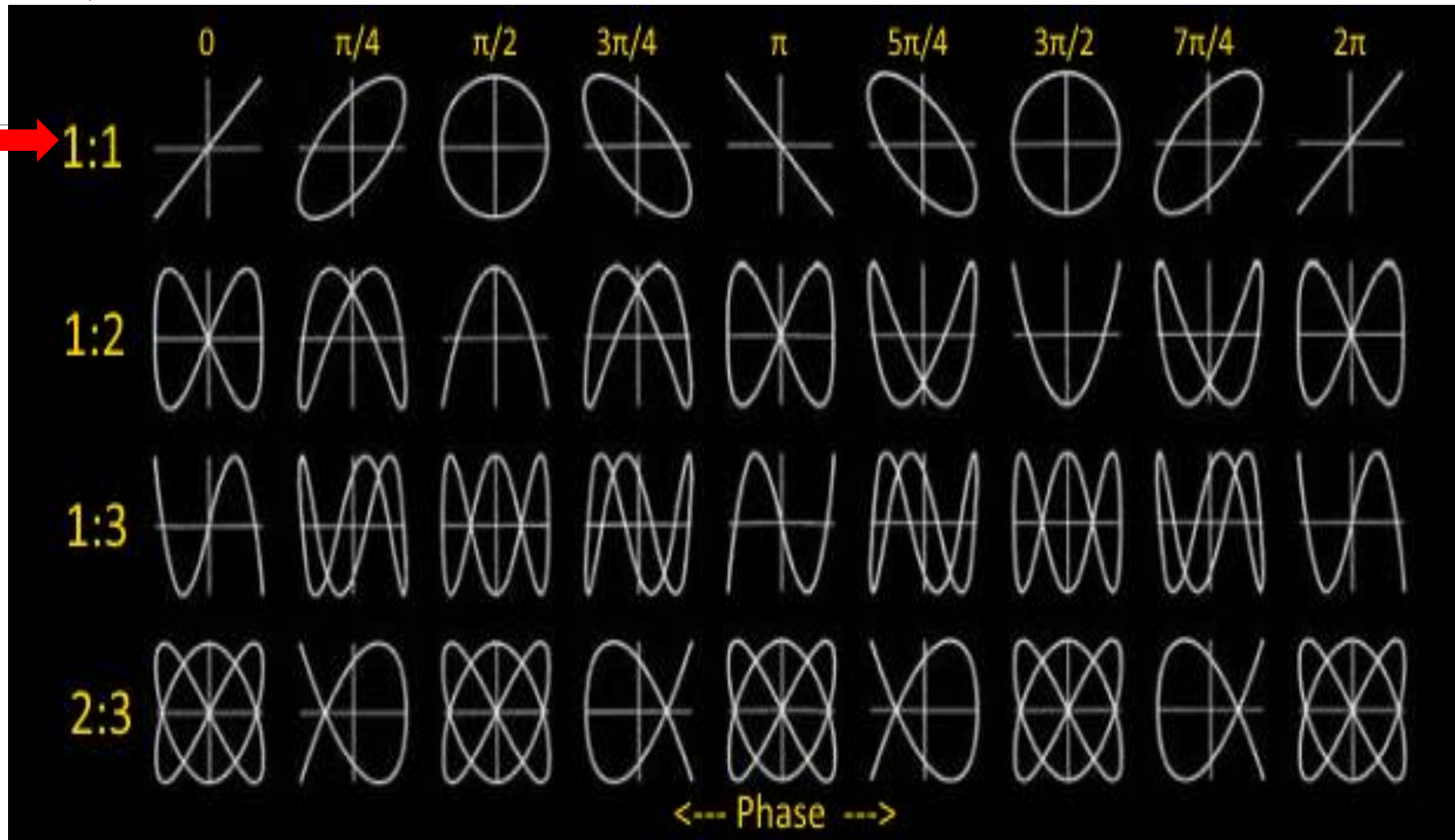
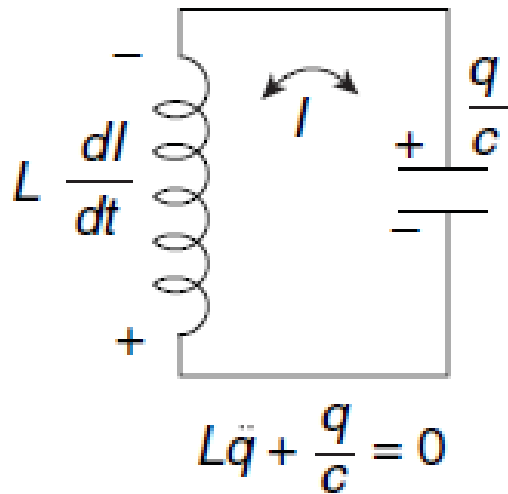http://www.uh.edu/engines/musicforawhile/musictext.htm

Lissajous figures

Frequency ratio

Identical frequencies in each component

http://datagenetics.com/blog/april22015/index.html

Simple harmonic oscillations in an electrical system

$$L\ddot{q} + \frac{q}{c} = 0$$

- Electrical system which oscillates simple harmonically.
- The sum of the voltage around the circuit is given by Kirchhoff's loop rule

$$V_L + V_c = 0$$

$$L\frac{dI}{dt} + \frac{q}{c} = 0$$

$$L\ddot{q} + \frac{q}{c} = 0$$

$$\ddot{q} + \omega^2 q = 0; \omega = 1/\sqrt{LC}$$

ω is considered as natural frequency of the circuit.

Comparison between the equations for mechanical and electrical oscillators

Mechanical (force) ➔ $m\ddot{x} + sx = 0$

Electrical (voltage) ➔ $L\ddot{q} + \dfrac{q}{C} = 0$

Mechanical (energy) ➔ $\dfrac{1}{2}m\ddot{x}^2 + \dfrac{1}{2}sx^2 = E$

Electrical (energy) ➔ $\dfrac{1}{2}L\ddot{q}^2 + \dfrac{1}{2}\dfrac{q^2}{C} = E$

Homework #1/2020

1. Show that $\displaystyle\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{2g}{l}}\frac{T_\alpha}{4}$ can be transformed to $\displaystyle T_\alpha = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}$

given that *(1)* $\cos\theta = 1 - 2sin^2\left(\frac{\theta}{2}\right)$, $\cos\alpha = 1 - 2sin^2\left(\frac{\alpha}{2}\right)$

(2) $sin\varphi = \dfrac{sin\frac{\theta}{2}}{sin\frac{\alpha}{2}}$ and *(3)* $k = sin\frac{\alpha}{2}$

2. Prove that $\dfrac{x^2}{A_1^2} + \dfrac{y^2}{A_2^2} - \dfrac{2xy}{A_1 A_2}\cos\delta = \sin^2\delta; \; \delta = \alpha_2 - \alpha_1$ can be derived from the superposition

of two perpendicular harmonic functions : $x = A_1\cos(\omega t + \alpha_1); y = A_2\cos(\omega t + \alpha_2)$